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A new class of realisations of the Lie algebra so$(q, 2n-q)$

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Abstract. The method for constructing boson realisations of semisimple Lie algebras formulated in our previous paper is applied to the case of so$(q, 2n-q)$, $q=2, 3, \ldots, n$. The realisations are expressed by means of certain recurrence formulae in terms of $(2n-2)$ boson pairs and generators of the subalgebra gl$(1,\mathbb{R})\oplus$so$(q-1, 2n-q-1)$. They are skew-Hermitian and Schurean.

1. Introduction

Canonical (boson) realisations of Lie algebras are used for studying physical systems with symmetries in the framework of the canonical formalism (see, e.g., Barut and Raczka 1977). They are especially useful in connection with the method of collective variables, e.g. in nuclear physics (Moshinsky 1984). Moreover, they play a role in purely mathematical investigations (e.g. in connection with the models for su(3) in terms of so$(n, 2)$ and so*$(2n)$ algebras (Le Blanc and Rowe 1986)).

In a recent paper (Burdík 1985) the method of constructing realisations for an arbitrary real semisimple algebra g was presented. It was shown that any induced representation can be rewritten as the so-called boson representation. The construction starts from a decomposition $g = n_+^b \oplus g_0^b \oplus n_-^b$ of g, which is a simple generalisation of the triangle decomposition (Dixmier 1984); it employs substantially induced representations (Zhelobenko and Stern 1983) of g with respect to a suitable representation σ of the subalgebra $g_0^b \oplus n_-^b$. It has been proved that the method gives realisations which possess two properties permitting their application in the representation theory. They are skew-Hermitian and Schurean.

In subsequent papers (Burdík 1986a, b) we have applied this method to the Lie algebras gl$(n+1, \mathbb{R})$ and sp$(n, \mathbb{R})$. In the case of the algebras gl$(n+1, \mathbb{R})$ we have constructed recurrence formulae which give realisations of gl$(n+1, \mathbb{R})$ in terms of $r(n+1-r)$ canonical pairs and generators of the subalgebra gl$(r, \mathbb{R})\oplus$gl$(n+1-r, \mathbb{R})$ for $r=1, 2, \ldots, n$. For the sp$(n, \mathbb{R})$ we have obtained recurrence formulae in terms of $r(2n-\frac{3}{2}r+\frac{1}{2})$ canonical pairs and generators of the subalgebra gl$(r, \mathbb{R})\oplus$sp$(n-r, \mathbb{R})$.

In the present paper, we apply the method of Burdík (1985) to the case of algebras so$(q, 2n-q)$ which are the real forms of the complex algebras so$(2n, \mathbb{C})$. We use the explicit forms of the triangle decompositions for the construction of this real algebra which we have previously constructed (Burdík 1986c). We obtain recurrence formulae

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which give realisations of $\mathrm{so}(q, 2n-q)$ in terms of $2n-2$ canonical pairs and generators of the subalgebra $\mathrm{gl}(1,\mathbb{R})\oplus\mathrm{so}(q-1, 2n-q-1)$. The resulting realisations are Schurean and skew-Hermitian. The calculation can be adapted easily for the algebras $\mathrm{so}(q, 2n+1-q)$ also.

The paper is organised as follows. All necessary prerequisites are listed in § 2 while § 3 contains the main results. Here the new wide families of realisations are derived. In § 4 the results are discussed and, in particular, a comparison is made with the realisations which were derived by Exner and Havlíček (1977).

2. Preliminaries

The Weyl algebra W_{2N} is the associative algebra over $\mathbb{C}$ with identity generated by $2N$ elements, p_i, q_i, where $i=1,2,\dots,N$, which satisfy the relations

$$[p_i, q_j]=\delta_{i,j} \qquad [p_i, p_j]=[q_i, q_j]=0 \tag{1}$$

for any $i, j=1,2,\dots,N$.

Let g, g_0 be real Lie algebras. By $\tilde{g}$, $\tilde{g}_0$ we denote their complexifications. Furthermore, $U(\tilde{g})$, $U(\tilde{g}_0)$ are the enveloping algebras of these complexifications.

Definition. A realisation of a Lie algebra g is a homomorphism

$$\tau : g \to W_{2N}\otimes U(\tilde{g}_0). \tag{2}$$

The homomorphism τ extends naturally to the homomorphic mapping (denoted by the same symbol τ) of the enveloping algebra $U(\tilde{g})$ into $W_{2N}\otimes U(\tilde{g}_0)$.

Definition. Let $Z(\tilde{g})$ be the centre of $U(\tilde{g})$. A realisation is called Schurean or Schur-realisation if all central elements $C\in Z(\tilde{g})$ are realised by $1\otimes C_0$ where the C_0 are central elements of the enveloping algebra $U(\tilde{g}_0)$.

In view of possible applications to the representation theory we introduce the involution '+' in W_{2N} by means of the following relations:

$$\begin{aligned} q_i^+ &= -q_i \\ p_i^+ &= p_i \end{aligned} \qquad \text{for } i=1,2,\dots,N. \tag{3a}$$

Similarly, the involution '+' on $U(\tilde{g}_0)$ is defined by

$$Y^+ = -Y \qquad \text{for } Y\in g_0. \tag{3b}$$

These involutions define naturally an involution on $W_{2N}\otimes U(\tilde{g}_0)$:

$$\left(\sum \alpha_j \Pi_j \otimes g_j\right)^+ = \sum \bar{\alpha}_j \Pi_j^+ \otimes g_j^+ \tag{3c}$$

where $\Pi_j\in W_{2N}$ and $g_j\in U(\tilde{g}_0)$.

Definition. Let g be a real Lie algebra and let '+' be the involution on $W_{2N}\otimes U(\tilde{g}_0)$ described above. A realisation τ of g on $W_{2N}\otimes U(\tilde{g}_0)$ is called skew-Hermitian if for all elements $X \in g$ the following relation holds:

$$(\tau(X))^{+} = -\tau(X). \tag{4}$$

The algebra $\mathrm{so}(2n, \mathbb{C})$ is the $n(2n-1)$-dimensional complex Lie algebra with the standard basis L_{ij}, $i, j = \pm 1, \pm 2, \ldots, \pm n$, the elements of which obey

$$L_{ij} = -L_{-j,-i} \tag{5}$$

and the commutation relations

$$[L_{ij}, L_{kl}] = \delta_{jk} L_{il} - \delta_{il} L_{kj} - \delta_{j,-l} L_{i,-k} + \delta_{i,-k} L_{-l,j}. \tag{6}$$

In appendix 1 we have specified an explicit form of the automorphisms which give the real forms of this algebra. Using these automorphisms we obtain for the algebras $\mathrm{so}(q, 2n-q)$ the following basis:

$$\begin{aligned} &L_{st} \\ &X_{\alpha\beta} = (L_{\alpha\beta} - L_{\beta\alpha}) \\ &Y_{\alpha\beta} = \mathrm{i}(L_{\alpha\beta} + L_{\beta\alpha}) \\ &X_{s\alpha} = (L_{s\alpha} + L_{s,-\alpha}) \\ &Y_{s\alpha} = \mathrm{i}(L_{s\alpha} - L_{s,-\alpha}) \end{aligned} \tag{7}$$

where $s, t = \pm 1, \pm 2, \ldots, \pm q$ and $\alpha, \beta = \pm(q+1), \pm(q+2), \ldots, \pm n$. The commutation relations in this basis are introduced in appendix 2.

For $b = L_{11}$ we define a decomposition of the algebra $\mathrm{so}(q, 2n-q)$ as

$$\begin{aligned} &g = n_{+}^{b} \oplus g_{0}^{b} \oplus n_{-}^{b} \\ &n_{+}^{b} = \mathbb{R}\{X \in g; [b, X] = \alpha_X X \text{ where } \alpha_X > 0\} \\ &g_{0}^{b} = \mathbb{R}\{X \in g; [b, X] = 0\} \\ &n_{-}^{b} = \mathbb{R}\{X \in g; [b, X] = -\alpha_X X \text{ where } \alpha_X > 0\}. \end{aligned} \tag{8}$$

We use these decompositions as a starting point for our construction (see also Burdík 1985, § 4).

3. Construction of realisations

Using the commutation relations (see appendix 2) we can bring the decomposition (8) into the form:

$$\begin{aligned} &n_{+}^{b} = \mathbb{R}\{L_{1i}, X_{1\alpha}, Y_{1\alpha}\} \\ &g_{0}^{b} = \mathbb{R}\{L_{11}, L_{ij}, X_{\alpha\beta}, Y_{\alpha\beta}, X_{i\alpha}, Y_{i\alpha}\} \\ &n_{-}^{b} = \mathbb{R}\{L_{i1}, X_{\alpha 1}, Y_{\alpha 1}\} \end{aligned} \tag{9}$$

where again $i, j = \pm 2, \pm 3, \ldots, \pm q$ and $\alpha, \beta = \pm(q+1), \pm(q+2), \ldots, \pm n$. The relation (5) implies that the basis of $\tilde{n}^b_+$ is formed by the following $(2n-2)$ elements:

$$\begin{vmatrix} L_{12} & L_{13} & \ldots & L_{1q} \\ L_{1,-2} & L_{1,-3} & \ldots & L_{1,-q} \\ X_{1,q+1} & X_{1,q+2} & \ldots & X_{1,n} \\ Y_{1,q+1} & X_{1,q+2} & \ldots & Y_{1,n} \end{vmatrix}. \tag{10}$$

We introduce an ordering in the above basis in which its elements are ordered lexicographically. The monomials of $U(\tilde{n}^b_+)$ can then be written as the matrices

$$\begin{vmatrix} n^L_2 & n^L_3 & \ldots & n^L_q \\ n^L_{-2} & n^L_{-3} & \ldots & n^L_{-q} \\ n^X_{q+1} & n^X_{q+2} & \ldots & n^X_n \\ n^X_{q+1} & n^Y_{q+2} & \ldots & n^Y_n \end{vmatrix}$$
$$= (L_{12}^{n^L_2}, \ldots, L_{1q}^{n^L_q})(L_{1,-2}^{n^L_{-2}}, \ldots, L_{1,-q}^{n^L_{-q}})(X_{1,q+1}^{n^X_{q+1}}, \ldots, X_{1,n}^{n^X_n})(Y_{1,q+1}^{n^Y_{q+1}}, \ldots, Y_{1n}^{n^Y_n}) \tag{11}$$

where of course n^L_i, n^X_α, n^Y_α belongs to N_0, the set of all non-negative integers.

Now we are able to apply the general construction described in Burdík (1985). Let σ be an auxiliary representation of the algebra $g^b_0 \oplus n^b_-$ on a vector space V such that

$$\begin{gathered} \sigma(n^b_-) = 0 \\ \sigma(g^b_0) \text{ is faithful.} \end{gathered} \tag{12}$$

We denote by W the carrier space of the induced representation $\rho = \mathrm{ind}(g, \sigma)$. If $v_1, \ldots, v_d$ is a basis in the space V, then the vectors

$$\begin{vmatrix} n^L_2 & \ldots & n^L_q \\ n^L_{-2} & \ldots & n^L_{-q} \\ n^X_{q+1} & \ldots & n^X_n \\ n^Y_{q+1} & \ldots & n^Y_n \end{vmatrix} \otimes v_i \tag{13}$$

form a basis W.

We define the creation and annihilation operators $\bar{a}^X_\alpha$, a^X_α on the space W in the following way:

$$\begin{gathered} \bar{a}^X_\alpha \begin{vmatrix} n^L_2 & \ldots & n^L_q \\ n^L_{-2} & \ldots & n^L_{-q} \\ n^X_{q+1} \ldots n^X_\alpha & \ldots & n^X_n \\ n^Y_{q+1} & \ldots & n^Y_n \end{vmatrix} \otimes v_i = \begin{vmatrix} n^L_2 & \ldots & n^L_q \\ n^L_{-2} & \ldots & n^L_{-q} \\ n^X_{q+1} \ldots & n^X_\alpha + 1 \ldots & n^X_n \\ n^Y_{q+1} & \ldots & n^Y_n \end{vmatrix} \otimes v_i \\ a^X_\alpha \begin{vmatrix} n^L_2 & \ldots & n^L_{-q} \\ n^L_{-2} & \ldots & n^L_{-q} \\ n^X_{q+1} \ldots n^X_\alpha & \ldots & n^X_n \\ n^Y_{q+1} & \ldots & n^Y_n \end{vmatrix} \otimes v_i = n^X_\alpha \begin{vmatrix} n^L_2 & \ldots & n^L_q \\ n^L_{-2} & \ldots & n^L_{-q} \\ n^X_{q+1} \ldots & n^X_\alpha - 1 \ldots & n^X_n \\ n^Y_{q+1} & \ldots & n^Y_n \end{vmatrix} \otimes v_i \end{gathered} \tag{14a}$$

and similarly we define the operators $\bar{a}^Y_\alpha$, a^Y_α, $\bar{a}^L_i$, a^L_i and $\bar{a}^L_{-i}$, a^L_{-i} for any $\alpha = (q+1), \ldots, n$; $i = 2, 3, \ldots, q$. For any $\alpha = (q+1), (q+2), \ldots,$ we put

$$\bar{a}^X_{-\alpha} = \bar{a}^X_\alpha \qquad \bar{a}^Y_{-\alpha} = -\bar{a}^Y_\alpha \qquad a^X_{-\alpha} = -a^X_\alpha \qquad a^Y_{-\alpha} = -a^Y_\alpha .$$

Furthermore we define the operators $\tilde{X}$ for any $X \in g_0^b$ by the relation

$$\tilde{X} = 1 \otimes \sigma(X). \tag{14b}$$

According to theorem 1 of Burdík (1985) the induced representation $\rho = \text{ind}(g, \sigma)$ can be rewritten using the operators (14a, b) defined above. We get the formulae

$$\begin{aligned}
\rho(L_{11}) &= \sum_{k=2}^{q} (\bar{a}_k^L a_k^L + \bar{a}_{-k}^L a_{-k}^L) + \sum_{\alpha=q+1}^{n} (\bar{a}_\alpha^X a_\alpha^X + \bar{a}_\alpha^Y a_\alpha^Y) + L_{11} \\
\rho(L_{ij}) &= -\bar{a}_j^L a_i^L + \bar{a}_{-i}^L a_j^L + L_{ij} \\
\rho(X_{\alpha\beta}) &= -\bar{a}_\beta^X a_\alpha^X + \bar{a}_\alpha^X a_\beta^X - \bar{a}_\beta^Y a_\alpha^Y + \bar{a}_\alpha^Y a_\beta^Y + X_{\alpha\beta} \\
\rho(Y_{\alpha\beta}) &= -\bar{a}_\beta^Y a_\alpha^X - \bar{a}_\alpha^Y a_\beta^X + \bar{a}_\beta^X a_\alpha^Y + \bar{a}_\alpha^X a_\beta^Y + Y_{\alpha\beta} \\
\rho(X_{i\alpha}) &= -\bar{a}_\alpha^X a_i^L + 2\bar{a}_{-i}^L a_\alpha^X + X_{i\alpha} \\
\rho(Y_{i\alpha}) &= -\bar{a}_\alpha^Y a_i^L + 2a_{-i}^L a_\alpha^Y + Y_{i\alpha}
\end{aligned} \tag{15}$$

where $i, j = \pm 2, \pm 3, \ldots, \pm q$ and $\alpha, \beta = \pm(q+1), \ldots, \pm n$ and further

$$\rho(L_{12}) = a_2^L$$

$$\begin{aligned}
\rho(L_{21}) = -\rho(L_{11})a_2^L &+ \sum_{k=2}^{q} \bar{a}_{-2}^L a_{-k}^L a_k^L + \sigma(L_{2k})a_k^L + \sigma(L_{2,-k})a_{-k}^L \\
&+ \sum_{\alpha=q+1}^{n} \sigma(X_{2\alpha})a_\alpha^X + \sigma(Y_{2\alpha})a_\alpha^Y - \bar{a}_{-2}^L(\bar{a}_\alpha^X a_\alpha^X + \bar{a}_\alpha^Y a_\alpha^Y).
\end{aligned}$$

We obtain the representation of the remaining generators using the commutation rules (see appendix 2).

Now the skew-Hermitian realisations sought are obtained easily by replacing the operators in the above expressions by suitable algebraic objects:

$$\begin{aligned}
\bar{a} &\to q \\
a &\to p \\
\sigma(X) &\to X.
\end{aligned} \tag{16}$$

For details, see Burdík (1985, § 3). They are given by the formulae

$$\begin{aligned}
\tau(L_{11}) &= \sum_{k=2}^{q} (q_k^L p_k^L + q_{-k}^L p_{-k}^L) + \sum_{\alpha=q+1}^{n} q_\alpha^X p_\alpha^X + q_\alpha^Y p_\alpha^Y + L_{11} + (n-1) \\
\tau(L_{ij}) &= -q_j^L p_i^L + q_{-i}^L p_{-j}^L + L_{ij} \\
\tau(X_{\alpha\beta}) &= -q_\beta^X p_\alpha^X + q_\alpha^X p_\beta^X - q_\beta^Y p_\alpha^Y + q_\alpha^Y p_\beta^Y + X_{\alpha\beta} \\
\tau(Y_{\alpha\beta}) &= -q_\beta^Y p_\alpha^X - q_\alpha^Y p_\beta^X + q_\beta^X p_\alpha^Y + q_\alpha^X p_\beta^Y + Y_{\alpha\beta} \\
\tau(X_{i\alpha}) &= -q_\alpha^X p_i^L + 2q_{-i}^L p_\alpha^X + X_{i\alpha} \\
\tau(Y_{i\alpha}) &= -q_\alpha^Y p_i^L + 2q_{-i}^L p_\alpha^Y + Y_{i\alpha}
\end{aligned} \tag{17}$$

where $i, j = \pm 2, \pm 3, \ldots, \pm q$; $\alpha = \pm(q+1), \pm(q+2), \ldots, \pm n$, and further

$$\tau(L_{12}) = q_2^L$$

$$\begin{aligned}
\tau(L_{21}) = -\tau(L_{11})p_2^L &+ \sum_{k=2}^{q} q_{-2}^L p_{-k}^L p_k^L + p_k^L L_{2k} + p_{-k}^L L_{2,-k} \\
&+ \sum_{\alpha=q+1}^{n} (X_{2\alpha})p_\alpha^X + (Y_{2\alpha})p_\alpha^Y - q_{-2}^L(p_\alpha^X p_\alpha^X + p_\alpha^Y p_\alpha^Y).
\end{aligned}$$

The element $b = L_{11}$ has the same meaning as the element b from Burdík (1985). Therefore, we can apply theorem 3 of that paper to the realisations (17) thus obtaining the following proposition.

Proposition. τ are Schur-realisations of $\mathrm{so}(q, 2n-q)$ in the $W_{2N} \otimes U(\mathrm{gl}(1, \mathbb{R}) \oplus \mathrm{so}(q-1, 2n-q-1))$.

4. Discussion

Explicit forms of realisations for $\mathrm{so}(n, 2)$ have been constructed by Le Blanc and Rowe (1986) using the method of coherent state representation. These realisations are defined by means of n canonical pairs and generators of a subalgebra $\mathrm{so}(n) \oplus \mathrm{so}(2)$. Another class of realisations has been described by Havlíček and Exner (1977); in their paper, realisations of $\mathrm{so}(m, n)$ are given in terms of $(m+n-2)$ canonical pairs and generators of $\mathrm{so}(m-1, n-1) \oplus \mathrm{gl}(1, \mathbb{R})$. Also the realisations given in the present paper are similar to those given explicitly in the papers mentioned, but we cannot give explicitly the relation which transforms these realisations from one to another.

Appendix 1

For any $q = 1, 2, \ldots, n$ we define linear mappings θ_q on $\tilde{g}$ in this way:

$$\theta_q(L_{st}) = -L_{ts}$$

$$\theta_q(L_{s,\alpha}) = -L_{-\alpha,s} \qquad \theta_q(L_{\alpha,s}) = -L_{s,-\alpha}$$

$$\theta_q(L_{\alpha,\beta}) = L_{\alpha\beta}$$

where $s, t = \pm 1, \pm 2, \ldots, \pm q$ and $\alpha, \beta = \pm(q+1), \ldots, \pm n$.

For $n = 2q+1$ odd we define further

$$\theta'(L_{st}) = L_{s+q_s,t+q_t}$$

$$\theta'(L_{s,t+q_t}) = L_{s+q_s,t} \qquad \theta'(L_{s+q_s,t}) = L_{s,t+q_t}$$

$$\theta'(L_{s,\alpha}) = -L_{s+q_s,\alpha} \qquad \theta'(L_{s+q_s,\alpha}) = L_{s,\alpha}$$

$$\theta'(L_{\alpha\alpha}) = L_{\alpha\alpha}$$

where $s, t = \pm 1, \pm 2, \ldots, \pm q$, $\alpha = \pm n$ and $q_s = q \operatorname{sgn} s$, and consequently for $n = 2q$ even we define

$$\theta''(L_{s,t}) = L_{s+q_s,t+q_t}$$

$$\theta''(L_{s,t+q_t}) = L_{s+q_s,t} \qquad \theta''(L_{s+q_s,t}) = L_{s,t+q_t}$$

where $s, t = \pm 1, \pm 2, \ldots, \pm q$.

Using the antilinear mapping

$$\psi(L_{ik}) = -L_{ki}$$

we can give for any linear mapping θ

$$g_\theta = \{X \in \tilde{g}, \theta \circ \psi(X) = X\}.$$

Theorem. The linear mappings θ', θ'', θ_q are Cartan automorphisms on $\tilde{g}$. Further, the algebras $g_{\theta'}$, $g_{\theta''}$ are isomorphic $\mathrm{so}^*(2n)$ and the algebra g_{θ_q} is isomorphic $\mathrm{so}(q, 2n-q)$ for any $q = 1, 2, \ldots, n$.

Proof. The proof is straightforward. See Helgasson (1962) for the method and Burdík (1986c) for detailed calculations in this case.

Appendix 2

Using the relations (6) we can compute commutation relations in the basis (7). In this appendix we give their explicit form:

$$[L_{ij}, L_{kl}] = \delta_{jk} L_{il} - \delta_{il} L_{kj} - \delta_{j,-l} L_{i,-k} + \delta_{i,-k} L_{-l,j}$$

$$[L_{ij}, X_{k\alpha}] = \delta_{jk} X_{i\alpha} - \delta_{i,-k} X_{-j,\alpha} \qquad [L_{ij}, X_{\alpha\beta}] = 0$$

$$[L_{ij}, Y_{k\alpha}] = \delta_{jk} Y_{i\alpha} + \delta_{i,-k} Y_{-j,\alpha} \qquad [L_{ij}, Y_{\alpha\beta}] = 0$$

$$[X_{i\alpha}, X_{j\beta}] = -\delta_{i,-j} X_{\alpha\beta} + \delta_{i,-j} X_{\alpha,-\beta} - 2(\delta_{\alpha\beta} + \delta_{\alpha,-\beta}) L_{i,-j}$$

$$[X_{i\alpha}, Y_{j\beta}] = -\delta_{i,-j} Y_{\alpha\beta} + \delta_{i,-j} Y_{\alpha,-\beta}$$

$$[Y_{i\alpha}, Y_{j\beta}] = -\delta_{i,-j} X_{\alpha\beta} + \delta_{i,-j} X_{\alpha,-\beta} - 2(\delta_{\alpha\beta} - \delta_{\alpha,-\beta}) L_{i,-j}$$

$$[X_{i\alpha}, X_{\beta\gamma}] = (\delta_{\alpha\beta} + \delta_{\alpha,-\beta}) X_{i\gamma} - (\delta_{\alpha\gamma} + \delta_{\alpha,-\gamma}) X_{i\beta}$$

$$[X_{i\alpha}, Y_{\beta\gamma}] = (\delta_{\alpha\beta} + \delta_{\alpha,-\beta}) Y_{i\gamma} + (\delta_{\alpha\gamma} + \delta_{-\alpha,\gamma}) Y_{i\beta}$$

$$[Y_{i\alpha}, Y_{\beta\gamma}] = -(\delta_{\alpha\beta} - \delta_{\alpha,-\beta}) X_{i\gamma} - (\delta_{\alpha\gamma} - \delta_{\alpha,-\gamma}) X_{i\beta}$$

$$[Y_{i\alpha}, X_{\beta\gamma}] = (\delta_{\alpha\beta} - \delta_{\alpha,-\beta}) Y_{i\gamma} - (\delta_{\alpha\gamma} - \delta_{\alpha,-\gamma}) Y_{i\beta}$$

$$[X_{\alpha\beta}, X_{\gamma\delta}] = \delta_{\beta\gamma} X_{\alpha\delta} - \delta_{\alpha\delta} X_{\gamma\beta} - \delta_{\beta,-\delta} X_{\alpha,-\gamma} + \delta_{\alpha,-j} X_{-\delta,\beta}$$

$$[X_{\alpha\beta}, Y_{\gamma\delta}] = \delta_{\beta\gamma} Y_{\alpha\delta} - \delta_{\alpha\delta} Y_{\gamma\beta} - \delta_{\beta,-\delta} Y_{\alpha,-\gamma} + \delta_{\alpha,-\gamma} X_{-\delta,\beta}$$

$$[Y_{\alpha\beta}, Y_{\gamma\delta}] = -\delta_{\beta\gamma} X_{\alpha\delta} + \delta_{\alpha\delta} X_{\gamma\beta} + \delta_{\beta,-\delta} X_{\alpha,-\gamma} - \delta_{\alpha,-\gamma} X_{-\delta,\beta}$$

where $i, j, k, l = \pm 1, \pm 2, \ldots, \pm q$ and $\alpha, \beta, \gamma, \delta = \pm(q+1), \pm(q+2), \ldots, \pm n$.

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